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# COLOURING STEINER QUADRUPLE SYSTEMS

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An algorithm is described which finds a proper 2-colouring of a Steiner quadruple system if one exists; the time required by this method is polynomial. By way of contrast, 2-colouring a *partial* Steiner quadruple system is shown to be NP-complete.

## 1. Introduction

A *hypergraph* is a pair $(V, E)$; $V$ is a $v$-set of *vertices* and $E$ is a set of subsets of $V$ called *edges*. A *graph* is a hypergraph $(V, E)$ in which each edge has cardinality 2. A $k$-uniform hypergraph is a hypergraph in which each edge has cardinality $k$.

A *Steiner quadruple system* (SQS) is a 4-uniform hypergraph $(V, E)$ having the additional property that each 3-subset of $V$ appears in precisely one edge (or *block*) of $E$. Steiner quadruple systems are *t-designs* with parameters $t=3$, $k=4$, $\lambda=1$; the construction of $t$-designs is the fundamental problem of combinatorial design theory [5], and finds application in the design of experiments [12] and the theory of error-correcting codes [8]. A *partial* Steiner quadruple system is a 4-uniform hypergraph $(V, E)$ having the additional property that each 3-subset of $V$ appears in *at most* one edge of $E$.

Colouring problems for graphs and hypergraphs arise in many areas of computer science [1, 3]. An *r-colouring* of a hypergraph is an assignment to each vertex of a colour chosen from a $r$-set of available colours; equivalently, it is a partition of the vertices into $r$ sets. An $r$-colouring is *proper* if no edge contains solely vertices of one colour. A hypergraph is *r-colourable* if it has a proper $r$-colouring, and is *r-chromatic* if it is $r$-colourable but not $(r-1)$-colourable. The *chromatic number* of a hypergraph $H$ is that $r$ for which $H$ is $r$-chromatic. Investigations of the chromatic number have led both directly and indirectly to elegant constructions for $t$-designs; one recent example is the investigation of 2-chromatic SQS [11].

Ever since the advent of computers, constructions for $t$-designs have been dis-

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covered by analyzing small known designs. Computational methods for the analysis of designs are naturally of interest here, in part to resolve the existence questions which remain open for small designs, and in part to suggest patterns for the construction of infinite families of designs.

We are concerned here with the problem of deciding whether a SQS is 2-chromatic; this problem arises in the work of Phelps and Rosa [9, 11]. Beyond the motivation from combinatorial design theory, we believe there is a further reason for interest. Determining whether a graph is 2-chromatic can be easily carried out in linear time - we need only decide if the graph is bipartite (see [1], for example). On the other hand, deciding whether a hypergraph is 2-chromatic is NP-complete [7]. We here prove a stronger result: deciding whether a partial SQS can be 2-coloured is NP-complete.

We require the following result due to Lovász [7].

**Theorem 1.** *Deciding whether a 3-uniform hypergraph is 2-colourable is NP-complete.*

**Proof.** Membership in NP is immediate. To show completeness, we give a polynomial time reduction from the problem of graph 3-colourability. Given a graph $G=(V,E)$, $V=\{v_1,\ldots,v_n\}$, we define a 3-uniform hypergraph $H=(W,F)$. The vertex set, $W$, is $\{\infty\}\cup\{x_{ij}\mid 1\le i\le n, 1\le j\le 3\}$. The edges in $F$ are

(1) $\{\infty,x_{ik},x_{jk}\}$ for all $\{v_i,v_j\}\in E$, $1\le k\le 3$.

(2) $\{x_{i1},x_{i2},x_{i3}\}$ for $1\le i\le n$.

Now $H$ is 2-colourable if and only if $G$ is 3-colourable. □

Using this result, we obtain

**Theorem 2.** *Deciding whether a partial SQS is 2-colourable is NP-complete.*

**Proof.** Membership in NP is immediate. We reduce 2-colourability of 3-uniform hypergraphs to our problem. We use the following gadget, called $R$. $R$ has vertex set $\{v_1,v_2,v_3\}\cup\{x_{ij},y_{ij}\mid 1\le i\le 3, 1\le j\le 9\}$. The edges of $R$ are

(1) $\{v_i,x_{ij},x_{ik},x_{im}\}$ and $\{v_i,y_{ij},y_{ik},y_{im}\}$, for $1\le i\le 3$, and $\{j,k,m\}$ one of $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{1,4,7\}$, $\{2,5,8\}$, $\{3,6,9\}$, $\{1,5,9\}$, $\{2,6,7\}$, $\{3,4,8\}$, $\{1,6,8\}$, $\{2,4,9\}$, $\{3,5,7\}$. (This set of twelve triples is the *affine Steiner triple system* of order 9.)

(2) $\{x_{1i},x_{1j},x_{2i},x_{3j}\}$ and $\{y_{3i},y_{3j},y_{1i},y_{2j}\}$ for all $1\le i,j\le 9$, $i\ne j$.

(3) $\{x_{2i},x_{3i},y_{1j},y_{2j}\}$ for $1\le i,j\le 9$, $i\ne j$.

Using this gadget $R$, we create a partial SQS $H$ from a 3-uniform hypergraph $G$. For every edge $\{a,b,c\}$ of $G$, we delete the edge and introduce a gadget $R$ by letting $\{a,b,c\}$ take the role of $\{v_1,v_2,v_3\}$ and introducing all new vertices for the remainder of $R$. $H$ is a partial SQS, and our transformation operates in polvnomial time. Thus we need only show that $H$ is 2-colourable if and only if $G$ is.

Suppose $G$ is 2-colourable. Then each edge $f=\{v_1, v_2, v_3\}$ is 2-coloured. Suppose that $v_i$ is coloured white, $v_j$ and $v_k$ black. Colour all vertices $\{x_{im}, y_{im} \mid 1 \leq m \leq 9\}$ black and all $\{x_{jm}, x_{km}, y_{jm}, y_{km} \mid 1 \leq m \leq 9\}$ white. This is a 2-colouring of $H$.

Before proving the converse, we remark that for the affine Steiner triple system of order 9, one must choose at least five vertices in order to ensure that every edge contains one of the chosen vertices.

Suppose there is a 2-colouring of $H$, but (to the contrary) there is no 2-colouring of $G$. Thus for every 2-colouring of $H$, there exists a set $f=\{v_1, v_2, v_3\}$ which is a monochromatic (1-coloured) edge in $G$. Suppose $\{v_1, v_2, v_3\}$ are all coloured white in $H$. In the 2-colouring of $H$, there are no monochromatic edges of type 1; hence, our earlier remark gives that at least five of the $\{x_{ij}\}$ and five of the $\{y_{ij}\}$ are black for each $i$. Similarly, no edge of type 2 can be all black; therefore, we may conclude that $x_{2i}$ and $x_{3i}$ have the same colour (for each $i$), *and* $y_{1j}$, $y_{2j}$ have the same colour (for each $j$). From this and the fact that at least five of $\{x_{2i}\}$ and five of the $\{y_{1j}\}$ are black, we conclude that there must be a monochromatic edge of type 3. This is the desired contradiction to the hypothetical 2-colouring of $H$. □

In the next section, we develop a polynomial time algorithm for deciding whether a SQS is 2-chromatic; this gives a vast improvement over the current best method for hypergraphs, although the problem is apparently not trivial as in the case of graphs.

## 2. A polynomial algorithm

We view a 2-colouring of a SQS $(V, E)$ as a partition of $V$ into two sets $V_1$ and $V_2$. It is proper if for any $e \in E$, $e \cap V_1 \neq e$ and $e \cap V_2 \neq e$. Our method uses the fact, proved by Doyen and Vandensavel [2], that if $\langle V_1, V_2\rangle$ is a proper 2-colouring of $(V, E)$, then $|V_1| = |V_2|$.

The algorithm will operate by extending a partial colouring, i.e. a partition of $V$ into three sets $V_1$, $V_2$ and $U$. Vertices in $V_1$ ($V_2$) have been assigned the first (second) colour; the colours of vertices in $U$ are as yet unspecified. A partial colouring $\langle V_1, V_2, U\rangle$ is *feasible* if there is a proper 2-colouring $\langle V_1', V_2'\rangle$ for which $V_1 \subseteq V_1'$ and $V_2 \subseteq V_2'$. All feasible partial colourings are proper, but of course the converse need not hold.

A simple-minded method which uses Doyen and Vandensavel's observation is the following. Given a partial colouring $\langle V_1, V_2, U\rangle$, first check that it is proper. If it is not, it is not feasible. Next check if $|V_1|$ or $|V_2|$ is $\frac{1}{2}|V|$; if so, we have completed a proper 2-colouring [2, 10]. In the final case, we attempt to extend the partial 2-colouring. For each $v \in U$ in turn, we determine whether $\langle V_1 \cup \{v\}, V_2, U - \{v\}\rangle$ is feasible. If any one of these is feasible, $\langle V_1, V_2, U\rangle$ is feasible; otherwise it is not.

Now a SQS $(V, E)$ is 2-colourable if and only if $\langle \emptyset, \emptyset, V \rangle$ is feasible. One serious defect with this method is that often we extend a partial colouring in such a way that the result is not proper. In particular, if $\{w, x, y, z\}$ is an edge for which $w, x, y \in V_1$ and $z \in U$, we note that the placement of $z$ in $V_1$ would result in an improper colouring. Hence, $z$ is placed in $V_2$.

With this in mind, we say $z$ is an *implicant* for $V_2$ $(V_1)$ if $z \in U$ and there is an edge $\{w, x, y, z\}$ with $w, x, y \in V_1$ $(V_2)$. The term 'implicant' suggests our idea – whenever three elements are coloured the same, we can guarantee that there is an edge $\{w, x, y, z\}$ for some $z$ and moreover, this $z$ *must* be assigned the other colour.

To circumvent the selection of vertices leading immediately to improper colourings, we introduce a process called *stabilization*. Given a partial colouring $\langle V_1, V_2, U \rangle$, we locate the set $U_1 \subseteq U$ of implicants for $V_1$ and the set $U_2 \subseteq U$ of implicants for $V_2$. If $U_1 \cap U_2 \neq \emptyset$, a proper colouring is impossible. In this event, the stabilization is said to *fail*. Otherwise, if $U_1 = U_2 = \emptyset$, stabilization is said to *succeed*. If none of these hold, we repeat the process and stabilize $\langle V_1 \cup U_1, V_2 \cup U_2, U - U_1 - U_2 \rangle$.

A *stable* partial colouring is one having no implicants. It is immediate from our earlier remarks that if $\langle V_1', V_2', U' \rangle$ is obtained by stabilizing $\langle V_1, V_2, U \rangle$ successfully, these two partial colourings are either both feasible or both infeasible.

Stabilization can be carried out in polynomial time, and thus it can be used to substantially improve the simple-minded algorithm mentioned earlier. After each selection, we stabilize the partial colouring and then attempt to extend the resulting partial colouring. In fact, we need only deal with stable partial colourings throughout.

To guarantee an improvement over the exponential running time of the simple-minded algorithm, we require a fact regarding stable partial colourings.

**Lemma 1.** *Let $\langle V_1, V_2, U \rangle$ be a stable partial colouring of a SQS $(V, E)$. Then* $|V_1| - 2 \leq |V_2| \leq |V_1| + 2$.

**Proof.** We first show $|V_2| \geq |V_1| - 2$. Let the elements of $V_1$ be $\{v_1, \ldots, v_r\}$. Consider the $r-2$ triples $\{v_1, v_2, v_i\}$, $3 \leq i \leq r$. Since $(V, E)$ is SQS, there is precisely one edge in $E$ containing the three vertices $v_1$, $v_2$ and $v_i$; this edge contains one more vertex which we denote $x_i$. Now for $i \neq j$, $x_i \neq x_j$ – otherwise $\{v_1, v_2, x_i\}$ would appear in more than one edge. In addition, each $x_i \in V_2$ since $\langle V_1, V_2, U \rangle$ is stable. But there are $|V_1| - 2$ $x_i$ and hence $|V_2| \geq |V_1| - 2$. By the same token, $|V_1| \geq |V_2| - 2$; this completes the proof. □

The stabilization with its obvious advantages corrects one defect of the simple-minded method, but there is another. The method tries *all* vertices from $U$ as candidates for inclusion in $V_1$. To profit from stabilization, we would naturally like to try only those vertices which result in a large number of implicants.

**Lemma 2.** *Suppose $\langle V_1, V_2, U \rangle$ is a stable partial colouring for a SQS $(V, E)$ with*

*$|V|=2n$. Let $r=|V_1|, s=|V_2|$. Then there is a constant $c$ and an element $w \in U$ for which*

*(1) $\langle V_1 \cup \{w\}, V_2, U-\{w\}\rangle$ contains at least $(r-1)(1-c/(n-r))$ implicants for $V_2$, and*

*(2) $\langle V_1, V_2 \cup \{w\}, U-\{w\}\rangle$ contains at least $(s-1)(1-c/(n-s))$ implicants for $V_1$.*

**Proof.** We deal here with the case when $r=s$; a similar argument when $r$ and $s$ differ by one or two results in only a different constant $c$. For $w \in U$, we classify four types of blocks of the form $\{w,x,y,z\}$; $a_1(w)$ denotes the number of such blocks with $x, y \in V_1$ and $z \in V_2$, $a_2(w)$ the number with $x, y \in V_1$ and $z \in U$, $b_1(w)$ the number with $x, y \in V_2$ and $z \in V_1$ and finally $b_2(w)$ the number with $x, y \in V_2$ and $z \in U$. Observe first that $a_1(w)+a_2(w)=b_1(w)+b_2(w)=\binom{r}{2}$, since $w$ appears with each pair of vertices in $V_1(V_2)$ exactly once. Furthermore,

$$\sum_{w \in U} a_1(w)=r(r-1).$$

This can be seen as follows. For each $v \in V_1$, we consider the set of triples of the form $\{\{x,y,z\} \mid \{v,x,y,z\} \in E\}$; this set is called the derived Steiner triple system (STS) for $v$ [10]. Now we claim that there are exactly $2(r-1)$ triples $\{x,y,z\}$ in this STS with $x \in V_1$, $y \in V_2$ and $z \in U$. This is easy counting, performed by observing that each pair of vertices in $V_1-\{v\}$ appears with exactly one vertex of $V_2$ in the derived STS (since the colouring is stable), and further, each pair of vertices from $V-\{v\}$ appears precisely once in a triple of the derived STS.

In this way, we obtain the contribution $2(r-1)$ to $\sum_{w \in U} a_1(w)$ from blocks containing a particular $v \in V_1$. Summing over all $v \in V_1$, we count each block twice; hence,

$$\sum_{w \in U} a_1(w)=r(r-1).$$

Similarly, we have

$$\sum_{w \in U} b_1(w)=r(r-1).$$

Then

$$\sum_{w \in U} (a_1(w)+b_1(w)) \le 2r(r-1).$$

Hence, there is a $w \in U$ for which

$$a_1(w)+b_1(w) \le \frac{2r(r-1)}{2n-2r}.$$

Recalling that

$$a_1(w)+a_2(w)=\binom{r}{2},$$

we have that

$$a_2(w) \geq \binom{r}{2} - \frac{r(r-1)}{n-r}.$$

Whence,

$$a_2(w) \geq \binom{r}{2}\left(1 - \frac{2}{n-r}\right).$$

Similarly,

$$b_2(w) \geq \binom{r}{2}\left(1 - \frac{2}{n-r}\right).$$

Consider the number of implicants for $V_2$ in $\langle V_1 \cup \{w\}, V_2, U - \{w\}\rangle$. Now $a_2(w)$ counts those blocks which yield an implicant for $V_2$, but not all are different. Recalling that each 3-subset appears exactly once, however, we obtain that the number of implicants for $V_2$ is at least

$$(r-1)\left(1 - \frac{2}{n-r}\right).$$

Symmetrically, we obtain that $\langle V_1, V_2 \cup \{w\}, U - \{w\}\rangle$ contains at least $(r-1)(1-2/(n-r))$ implicants for $V_1$.

Analysis of this case, $r=s$, satisfies the statement of the theorem with constant $c=2$. The remaining two cases also satisfy the statement of the theorem, with different constants. □

Together with stabilization, we introduce a greedy selection process into the algorithm; let us state a complete algorithm which tests whether a partial colouring for a SQS $(V, E)$ with $2n$ vertices is feasible.

**Procedure** COLOUR($\langle V_1, V_2, U\rangle$)
  if $\langle V_1, V_2, U\rangle$ is not proper, return with failure.
  stabilize $\langle V_1, V_2, U\rangle$.
  if stabilization fails, return with failure.
  if $|V_1| = n$ or $|V_2| = n$, return with success.
  *comment:* now make greedy selection.
  select $w \in U$, which maximizes the number of implicants in $\langle V_1 \cup \{w\}, V_2, U - \{w\}\rangle$ *and* in $\langle V_1, V_2 \cup \{w\}, U - \{w\}\rangle$.
  if COLOUR($\langle V_1 \cup \{w\}, V_2, U - \{w\}\rangle$) succeeds, return with success.
  else if COLOUR($\langle V_1, V_2 \cup \{w\}, U - \{w\}\rangle$) succeeds, return with success.
  else return with failure.
**end** COLOUR.

**Lemma 3.** COLOUR($\langle \emptyset, \emptyset, V\rangle$) *succeeds on a SQS* $(V, E)$ *if and only if it is 2-colourable.*

**Proof.** In the event of success, COLOUR actually finds a proper 2-colouring. The other direction follows from Doyen and Vandensavel's observation and Lemma 2. □

Of course, procedure COLOUR was designed in a manner which essentially guarantees its correctness *a priori*. The interesting issue here is the timing.

**Lemma 4.** *On a SQS* $(V, E)$ *with* $|V| = 2n$, COLOUR$(\langle\emptyset, \emptyset, V\rangle)$ *completes its work in polynomial time.*

**Proof.** Each invocation of COLOUR carries out a polynomial amount of work. Thus we need only show that the depth of recursion is $O(\log n)$, since an invocation of COLOUR invokes itself at most twice. Suppose one invocation of COLOUR obtains the partial colouring $\langle V_1, V_2, U\rangle$ after successful stabilization, and suppose that $r = |V_1|$.

We first show that, for $r \le \frac{1}{2}n$ and $r \ge c'$ for $c'$ a constant, there is a constant $k > 1$ for which if $m = |V_1| + |V_2|$, then in the next invocation of COLOUR after stabilization at least $km$ vertices are coloured. But from Lemma 2 and $r \le \frac{1}{2}n$, we have that at least $r - c - 1$ new vertices are coloured in the stabilization performed by the next invocation of COLOUR. In fact, using Lemma 1 we can make a stronger statement. Since at least $r - c - 1$ vertices are added to $V_2$ by stabilization, at least $r - c - 5$ are added to $V_1$ by stabilization. But then for $\frac{1}{2}n \ge r \ge 2c + 6$, we always have that if $m$ vertices were coloured, at least $\frac{3}{2}m$ vertices will be coloured in the next invocation of COLOUR.

Now consider $r > \frac{1}{2}n$. The number of implicants for a colour cannot exceed the number of vertices which are yet to be assigned that colour. Then

$$n - r \ge r\left(1 - \frac{c}{n - r}\right),$$

whence

$$2r^2 - (3n - c)r + n^2 \ge 0.$$

Thus either

$$r \ge \tfrac{1}{4}(3n - c + \sqrt{n^2 - 6cn + c^2})$$

or

$$r \le \tfrac{1}{4}(3n - c - \sqrt{n^2 - 6cn + c^2}).$$

We conclude that there are constants $c_1, c_2$ for which $r < \frac{1}{2}n + c_1$ or $r > n - c_2$, and hence once $r > \frac{1}{2}n$ only a constant number of steps are required to complete the colouring.

We conclude that the depth of recursion is $O(\log n)$, which completes the proof. □

Lemmas 3 and 4 together yield

**Theorem 3.** *Deciding whether a Steiner quadruple system on n vertices is 2-colourable can be performed in polynomial time.*

## 3. Conclusions

Colouring methods for $t$-designs are important for the development of new construction techniques, and they contribute thereby to the ongoing effort in resolving existence questions for $t$-designs. The method developed here for recognizing 2-chromatic Steiner quadruple systems relies completely on the presence of implicants, which in turn requires the '2 colour' restriction. Furthermore, our NP-completeness result for 2-colouring a partial Steiner quadruple system shows that implicants rely heavily on the constraint on 3-subsets in SQS.

Whenever we are 2-colouring a family of $t$-designs for which similar results on implicants hold, our methods again yield a polynomial algorithm. As an aside, note that particular instances of this arise for $t$-designs with $k=t+1$, $\lambda \geq 1$.

We remark in conclusion that the NP-completeness of 2-colouring a partial SQS and the polynomial time algorithm for 2-colouring a SQS suggest an interesting combinatorial problem. Combinatorial design theorists have investigated *embedding problems,* the completion of a partial system to a design by the addition of further vertices and blocks [6]. Many embedding methods are known which are efficiently computable.

An efficiently computable embedding which creates a 2-chromatic SQS from a 2-chromatic partial SQS would establish that P = NP. Thus, combinatorial design theorists will likely not find such an embedding. One can ask for a weaker type of embedding: given the 2-colouring, find a technique which efficiently embeds a 2-coloured partial SQS into a 2-coloured SQS. Alternatively, one might find an embedding method which embeds a 2-chromatic partial SQS into a 2-chromatic SQS, where the sizes are polynomially related but computation of the embedding employs exponential time. In either case, new embedding techniques would have to be developed.

Another algorithmic problem on Steiner quadruple systems is suggested by our results. Our method for recognizing 2-chromatic SQS also gives an algorithm for deciding if a SQS of order $2n$ has a maximum independent set of size $n$, which is the largest possible [2, 10]. An interesting open question is to determine the complexity of deciding whether a SQS has an independent set of size at least $k$.

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